A WARPED PRODUCT VERSION OF THE CHEEGER-GROMOLL SPLITTING THEOREM

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Abstract. We prove a new generalization of the Cheeger-Gromoll splitting theorem where we obtain a warped product splitting under the existence of a line. The curvature condition in our splitting is a curvature dimension inequality of the form $CD(0,1)$. Even though we have to allow warping in our splitting, we are able to recover topological applications. In particular, for a smooth compact Riemannian manifold admitting a density which is $CD(0,1)$, we show that the fundamental group of M is the fundamental group of a compact manifold with nonnegative sectional curvature. If the space is also locally homogeneous, we obtain that the space also admits a metric of non-negative sectional curvature. Both of these obstructions give many examples of Riemannian metrics which do not admit any smooth density which is $CD(0,1)$.

1. Introduction

The Cheeger-Gromoll splitting theorem states that a complete manifold with non-negative Ricci curvature that admits a line is isometric to a product metric of the form $\mathbb{R} \times L$. A *line* is a geodesic $\gamma : (-\infty, \infty) \to M$ which is minimizing between any two points on γ. A simple way to construct a space with a line that is not isometric to a product is to take the topological product $\mathbb{R} \times L$ with metric $g = dr^2 + g_r$ where g_r with $r \in (-\infty, \infty)$ is a smooth one-parameter family of smooth metrics on L. The splitting theorem implies that any such complete metric g has non-negative Ricci curvature if and only if $g_r = g_0$ and g_0 has non-negative Ricci curvature.

In this paper we give a generalization of the splitting theorem which characterizes a more general class of spaces. If there is a positive function $u(r)$ on $\mathbb{R}$ such that $g = dr^2 + u^2(r)g_0$ for a fixed metric g_0 then g is called a *warped product over* $\mathbb{R}$. Note that such metrics always contains a line in the $\mathbb{R}$ direction. The curvature condition for our splitting theorem is a curvature dimension inequality, which is a generalization of a lower bound on Ricci curvature.

Definition 1.1. Let (M^n, g) be a Riemannian manifold and f a smooth real valued function on M. The N-dimensional generalized Ricci tensor of the triple (M, g, f) is

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{df \otimes df}{N-n}.$$

We say that (M, g, f) is $\mathrm{CD}(\lambda, N)$, ($\lambda \in \mathbb{R}, N \in (-\infty, \infty]$) if $\mathrm{Ric}_f^N \geq \lambda$.

If (M, g) has $\mathrm{Ric} \geq \lambda$ then taking f to be constant, we will have $CD(\lambda, N)$ for all N. Until recently, the study of curvature dimension inequalities has focused on the cases $N > n$ or $N = \infty$. However, there is an emerging body of research for the more general condition where $N < n$. The first systematic investigations of the

range $N < n$, [Oht, KM] appeared almost simultaneously. [Oht] has shown that the $CD(K, N)$ condition $N < 0$ is characterized by convexity properties of a relative entropy. In particular, this allows one to make sense of the $CD(K, N)$, $N < 0$ condition on non-smooth spaces, also see the earlier works of Ohta-Takatsu[OT11, OT13]. Kolesnikov-Milman [KM] and Milman [Mila] have also studied isoperimetric, functional, and concentration properties of spaces satisfying $CD(\lambda, N)$, $N < 1$. Also see Klartag [Kla]. For interesting examples of $CD(K, N)$ densities on the sphere, see [Milb]. As is pointed out by Milman, the study of curvature dimension inequalities on Euclidean space with $N < 0$ was investigated in the 1970s by Borell [Bor74] and Brascamp-Lieb[BL76]. We should also caution the reader that our definition of $CD(\lambda, N)$, which matches [Mila], is not equivalent to the definition given by Bakry-Emery [BÉ85] in the range $N \in [0, n)$, see [Mila, Section 7.5].

We will add to these works by generalizing the splitting theorem to the $CD(0, N)$ condition where $N \leq 1$. The first results for weighted Ricci curvature were proven by Lichnerowicz in [Lic70, Lic71]. One of his results (in our notation) is that if there is a bounded function f such that (M, g, f) is $CD(0, \infty)$ then the Cheeger-Gromoll splitting theorem holds. Since hyperbolic space admits an unbounded density which is $CD(0, \infty)$, the assumption that f be bounded is necessary. Fang-Li-Zhang [FLZ09] also showed that the splitting theorem holds for $CD(0, N)$, $N > n$ with no assumptions on f, and improve Lichnerowicz's result for $CD(0, \infty)$ by only assuming an upper bound on f. The splitting theorem for non-smooth spaces in the case $N > 1$ has also recently been proven by Gigli [Gig14, Gig].

We will show that Lichnerowicz's smooth splitting theorem holds for the weaker $CD(0, N)$ condition where $N < 1$. On the other hand, the theorem is not true when $N = 1$ as there are warped product spaces that admit a bounded function f so that the space is $CD(0, 1)$. Our main result says that these are the only such examples.

Theorem 1.2. *Suppose that a complete Riemannian manifold (M, g) admits a line and a function f which is bounded above which is $CD(0, 1)$, then (M, g) is a warped product over $\mathbb{R}$.*

As a corollary of the proof of Theorem 1.2 we obtain the isometric product splitting for $CD(0, N)$ with $N < 1$.

Corollary 1.3. *Suppose that (M, g) is a complete Riemannian manifold that admits a line and a function f which is bounded above which is $CD(0, N)$ for some $N < 1$, then M splits isometrically as a product $\mathbb{R} \times L$ and f is a function on L only.*

Remark 1.4. We actually prove versions of Theorem 1.2 and Corollary 1.3 that are more general in two ways. The first is that we can weaken the upper bound on f assumption to an integral condition along geodesics that we call f-completeness. This condition turns out to be equivalent to the completeness of a certain weighted affine connection, see [WY] for further study in this direction. Secondly we have versions where the function f can be replaced with a vector field X. We also prove a version of the splitting theorem for manifolds with boundary. We delay discussing these results until Sections 5 and 6.

By applying the Cheeger-Gromoll splitting theorem iteratively one can show that a complete non-compact manifold of non-negative Ricci curvature is isometric

to a product of a Euclidean space and a space with no lines. We also obtain a sharp structure theorem for spaces which are $CD(0,1)$ with f bounded above. We obtain a topological splitting $M = \mathbb{R}^k \times N$, but the metric is a warped product $g = dr^2 + u^2(r)\,(g_{\mathbb{R}^{k-1}} + g_N)$, where g_N is a metric with no lines. See Theorem 4.1 below for the precise statement.

Despite the weaker geometric splitting, we are able to recover the classical applications to topology and homogeneous spaces of the Cheeger-Gromoll splitting theorem. For example, an obvious corollary is that if (M, g, f) is $CD(0,1)$ and f is bounded above then M has at most two ends, and only one end if there is a point with $\mathrm{Ric}_f^1 > 0$.

Another topological result comes from applying the splitting theorem to the universal cover of a compact (M, g, f) which is $CD(0,1)$. When equipped with the pullback of f and g, the universal cover will also be $CD(0,1)$ and the potential function will be bounded. In this situation we obtain a sharp geometric structure theorem for the universal cover, see Theorem 4.5 for the precise statement. Using a result of Wilking [Wil00] along with the arguments of Cheeger-Gromoll we obtain the following statement about the topology of compact $CD(0,1)$ spaces.

Theorem 1.5. *Suppose that (M, g, f) is $CD(0,1)$ with M compact. Then,*

(1) *$\pi_1(M)$ is the fundamental group of a compact manifold with nonnegative sectional curvature.*
(2) *$b_1(M) \leq n$ and $b_1(M) = n$ if and only if M is isometric to a flat manifold and f is constant.*
(3) *If moreover, there is a point where $\mathrm{Ric}_f^1 > 0$, then $\pi_1(M)$ is finite.*

This result gives many examples of compact Riemannian manifolds which do not support any function f which is $CD(0,1)$. In fact, it is an open question whether there is a topological difference between spaces which are $CD(0,N)$ and spaces of non-negative Ricci curvature. That is, we have the following question: if (M, g, f) is $CD(0,N)$ does M also support a metric with non-negative Ricci curvature? [KW, Proposition 3.7] implies this is true if (M, g) is compact and homogeneous and, in fact, g must have non-negative Ricci curvature. Using the splitting theorem we also obtain a complete classification to compact locally homogeneous spaces which are $CD(0,1)$.

Theorem 1.6. *Suppose (M, g, f) is $CD(0,1)$ where (M, g) is a compact locally homogeneous space. Then M is a flat bundle over a compact locally homogeneous space of non-negative Ricci curvature. In particular, M admits a (possibly different) invariant metric of nonnegative sectional curvature.*

Although Milman [Mila] has obtained information about spaces which are $CD(0,N)$ with $N < 1$, Theorem 1.2 appears novel as it seems to be the first result in the literature for the case $N = 1$. The main new ingredient of the proof of our splitting theorem is a new Bochner type formula which we use to obtain a new Laplacian comparison theorem for a $CD(0,1)$ space. Our Bochner formula generalizes the classical one for Ricci curvature in a different way than the Bochner formulas of Lichnerowicz [Lic70, Lic71] Bakry-Emery [BÉ85, Bak94], and Ohta [Oht] for $CD(0,N)$ in the $N = \infty$, $N > n$, and $N < 0$ cases respectively. Under the $CD(0,1)$ assumption, our formula only applies to distance functions, but it gives a philosophical connection between Bakry-Emery's definition of curvature dimension and the results in [Mila, KM]. Further applications of the Bochner formula are developed in [WY].

One way in which to summarize our results is to say that $N = 1$ is a critical parameter for the splitting theorem where the isometric splitting theorem fails but is replaced by the weaker warped product splitting. A natural question is whether warped product splitting holds for $N \in (1, n)$. Our methods do not seem to say anything in this case.

After the completion of this paper, some similar rigidity theorems for Bakry-Emery Ricci tensors for Lorentzian manifolds have also been established in [WW16]. The same limited loss of rigidity from an isometric product to a warped product is also found in that case.

We would also like to point out that the intuition that led us to consider that $N = 1$ might be a critical dimension, came from recent work of the author that defines a notion of sectional curvature for manifolds with density [Wyl15]. In that work, we develop a notion of weighted sectional curvature, which we called $\overline{\sec}_f$. It comes up from considering modifying the radial curvature equation applied to Jacobi fields. The average of curvatures $\overline{\sec}_f$ over an orthonormal basis is Ric_f^1 in the same way that the sectional curvatures average to the Ricci curvature. Some of the examples in the next section arise in [KW] in the context of studying weighted sectional curvatures and our new Bochner formula can be derived from tracing some of the equations in [Wyl15].

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2. Twisted and Warped Products over $\mathbb{R}$

In this section we discuss the examples that arise in our splitting theorem and also show that Lichneorwicz splitting theorem does not hold for $CD(0, 1)$. We have to initially consider spaces which are slightly more general than a warped product. Let (L, h_L) be an $(n-1)$-dimensional Riemannian manifold, let $M = \mathbb{R} \times L$, and let $\psi(r, p)$ be an arbitrary real valued function on M. A *twisted product* metric over $\mathbb{R}$ is a metric g_M of the form

$$g_M = dr^2 + e^{\frac{2\psi}{n-1}} h_L \qquad r \in (-\infty, \infty).$$

Twisted products always contain a line given by the geodesic $\gamma(t) = (t, x_0)$, where x_0 is a fixed point in L. If ψ is a function of r only, then the metric g_M is called a *warped product*.

The connection and Ricci tensor of a twisted product, can be found for example as a special case of the equations in, [FLGRKÜ01].

Proposition 2.1. *Let $g_M = dr^2 + e^{\frac{2\psi}{n-1}} h_L$ and let U and V be vector fields on L. Then the Riemannian connection of g_M is given by*

$$\begin{aligned} \nabla_{\frac{\partial}{\partial r}} V &= \frac{1}{(n-1)} \frac{\partial \psi}{\partial r} V \\ \nabla_U V &= \nabla^L_U V + \frac{1}{n-1} \left(D_U(\psi) V + D_V(\psi) U - g_M(U, V) \nabla \psi \right). \end{aligned}$$

The Ricci tensor is given by

$$\begin{aligned}
\mathrm{Ric}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) &= -\frac{\partial^2\psi}{\partial r^2}-\frac{1}{n-1}\left(\frac{\partial\psi}{\partial r}\right)^2\\
\mathrm{Ric}\left(\frac{\partial}{\partial r},V\right) &= \frac{(2-n)}{n-1}D_V D_{\frac{\partial}{\partial r}}\psi\\
\mathrm{Ric}\left(U,V\right) &= \mathrm{Ric}_{h_L}\left(U,V\right)+\left(\frac{1}{n-1}\right)\mathrm{Hess}\psi\left(U,V\right)+D_U D_V(\psi)-D_{\nabla^L_U V}(\psi)\\
&\quad+\frac{1}{n-1}D_U(\psi)D_V(\psi)-\frac{1}{n-1}\left(\Delta\psi+\frac{|\nabla\psi|^2}{n-1}\right)g_M(U,V).
\end{aligned}$$

A natural choice for the potential function f is $f=\psi$, since then it follows from the equations above that $\mathrm{Ric}^1_f\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)=0$. On the other hand, we can also show that if the potential $f=\psi$ is $CD(0,1)$ then the metric can be written as a warped product.

Proposition 2.2. *Suppose that (M,g_M,f) is $CD(0,1)$ with the metric of the form $g_M=dr^2+e^{\frac{2f}{n-1}}h_L$, then $f=\phi(r)+f_L(x)$, where $\phi:\mathbb{R}\to\mathbb{R}$ and $f_L:L\to\mathbb{R}$. In particular, the metric g_M is a warped product of the form $g_M=dr^2+e^{\frac{2\phi(r)}{n-1}}g_L$ where $g_L=e^{\frac{2f_L(x)}{n-1}}h_L$.*

Proof. Since $\mathrm{Ric}^1_f\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)=0$ and $\mathrm{Ric}^1_f\geq 0$ we must have that that $\mathrm{Ric}^1_f\left(\frac{\partial}{\partial r},V\right)=0$ for all $V\perp\frac{\partial}{\partial r}$.

Fix a point x in L, let $\frac{\partial}{\partial y^i}$, $i=1,\dots,n-1$ be an orthonormal basis of local coordinates around x in the h_L metric. Write $\nabla f=a(r,y)\frac{\partial}{\partial r}+b_i(r,y)\frac{\partial}{\partial y^i}$, then $a=\frac{\partial f}{\partial r}$ and $b_i=e^{\frac{-2f}{n-1}}\frac{\partial f}{\partial y^i}$.

Then we have

$$\begin{aligned}
\mathrm{Hess}f\left(\frac{\partial}{\partial r},\frac{\partial}{\partial y^k}\right) &= g_M\left(\nabla_{\frac{\partial}{\partial r}}\nabla f,\frac{\partial}{\partial y^k}\right)\\
&= \sum_{i=1}^{n-1}g_M\left(\nabla_{\frac{\partial}{\partial r}}\left(e^{\frac{-2f}{n-1}}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^i}\right),\frac{\partial}{\partial y^k}\right)\\
&= \frac{\partial}{\partial r}\left(e^{\frac{-2f}{n-1}}\frac{\partial f}{\partial y^k}\right)e^{\frac{2f}{n-1}}+\frac{1}{n-1}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial r}\\
&= \frac{\partial}{\partial r}\frac{\partial}{\partial y^k}(f)-\frac{1}{n-1}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial r}.
\end{aligned}$$

This combined with Proposition 2.1 implies that

$$\begin{aligned}
0 &= \mathrm{Ric}^1_f\left(\frac{\partial}{\partial r},\frac{\partial}{\partial y^k}\right)\\
&= \frac{2-n}{n-1}\frac{\partial}{\partial y^k}\frac{\partial}{\partial r}(f)+\frac{\partial}{\partial r}\frac{\partial}{\partial y^k}(f)-\frac{1}{n-1}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial r}+\frac{1}{n-1}\frac{\partial f}{\partial y^k}\frac{\partial f}{\partial r}\\
&= \left(1+\frac{2-n}{n-1}\right)\frac{\partial}{\partial r}\frac{\partial}{\partial y^k}(f)\\
&= \frac{1}{n-1}\frac{\partial}{\partial r}\frac{\partial}{\partial y^k}(f)=\frac{1}{n-1}\frac{\partial}{\partial y^k}\frac{\partial}{\partial r}(f)
\end{aligned}$$

This implies that $\frac{\partial f}{\partial r}$ is constant in directions tangent to L and $\frac{\partial f}{\partial y^k}$ is constant in the r direction. As in [FLGRKÜ01, Theorem 1] this implies that $f = \phi(r) + f_L$ where $f_L : L \to \mathbb{R}$. Then

$$g_M = dr^2 + e^{\frac{2f}{n-1}} h_L = dr^2 + e^{\frac{2\phi(r)}{n-1}} \left(e^{\frac{2f_L(x)}{n-1}} h_L \right)$$

which gives the result.

□

The triples (M, g, f) of the form given by the conclusion of Proposition 2.2 are exactly the spaces that arise in our splitting theorem. To aid our exposition we will call these triples *split spaces*. That is (M, g_M, f) is a split space if M is diffeomorphic to $\mathbb{R} \times L$, $f = \phi(r) + f_L(x)$, where $\phi : \mathbb{R} \to \mathbb{R}$ and $f_L : L \to \mathbb{R}$, and $g_M = dr^2 + e^{\frac{2\phi(r)}{n-1}} g_L$ for a fixed metric g_L on L. From the calculations above we have that $\mathrm{Ric}_f^1 \left(\frac{\partial}{\partial r}, \cdot \right) = 0$ for any split space.

In order for a split space to be $CD(0,1)$ we need an additional curvature assumption for the triple (L, g_L, f_L). In considering what this condition should be, note that it is not true that the isometric product of spaces which are $CD(0,1)$ are $CD(0,1)$. This is because the definition of the curvature dimension condition depends on the dimension of the manifold. In fact, the product metric $L^{n-k} \times \mathbb{R}^k$ with a potential function f defined on L admits $CD(0,1)$ if and only if (L^{n-k}, g_L, f) is $CD(0, 1-k)$. This motivates the following proposition.

Proposition 2.3. *A split space (M, g_M, f) is $CD(0,1)$ if and only if*

$$(\mathrm{Ric}_{g_L})^0_{f_L} \geq \sup_r \left(\frac{1}{n-1} \frac{\partial^2 \phi}{\partial r^2} e^{\frac{2\phi}{n-1}} \right) g_L$$

In particular, if (M, g_M, f) is $CD(0,1)$ then (L, g_L, f_L) is $CD(0,0)$.

Proof. From Proposition 2.2 we already see that $\mathrm{Ric}_f^1 \left(\frac{\partial}{\partial r}, Y \right) = 0$ for all vector fields Y, so we just need to consider $\mathrm{Ric}_f^1 (U, V)$ for $U, V \perp \frac{\partial}{\partial r}$. From Proposition 2.1 we have that

$$\begin{aligned} \mathrm{Hess}\phi(U,V) &= \frac{1}{n-1} \left(\frac{\partial \phi}{\partial r} \right)^2 g_M(U,V) \\ \Delta \phi &= \frac{\partial^2 \phi}{\partial r^2} + \left(\frac{\partial \phi}{\partial r} \right)^2 \end{aligned}$$

and thus

$$\mathrm{Ric}\,(U,V) = \mathrm{Ric}_{g_L}(U,V) - \frac{1}{n-1} \left(\frac{\partial^2 \phi}{\partial r^2} + \left(\frac{\partial \phi}{\partial r} \right)^2 \right) g_M(U,V).$$

Moreover,

$$\mathrm{Hess} f(U,V) = \mathrm{Hess}\phi(U,V) + \mathrm{Hess}^L f_L(U,V) = \frac{1}{n-1} \left(\frac{\partial \phi}{\partial r} \right)^2 g_M(U,V) + \mathrm{Hess}^L f_L(U,V).$$

So

$$\mathrm{Ric}_f^1(U,V) = (\mathrm{Ric}_{g_L})^0_{f_L}(U,V) - \frac{1}{n-1} \left(\frac{\partial^2 \phi}{\partial r^2} \right) g_M(U,V).$$

Which gives the first part of the result.

Now suppose that (L, g_L, f_L) is not $CD(0,0)$, then there is a constant $a > 0$ such that

$$\frac{1}{n-1}\frac{\partial^2 \phi}{\partial r^2} \leq -ae^{\frac{-2\phi}{n-1}}$$

Letting $y = \phi/(n-1)$ we have $y'' \leq -ae^{-2y}$. Solutions to this inequality can be bounded above by the appropriate solutions to the equation $v'' = -ae^{-2v}$. This equation can be solved explicitly and we can see that all solutions v go to $-\infty$ in finite time, but this contradicts that ϕ is defined for all r. $\square$

Now we can construct the examples of spaces with bounded f and containing a line which do not split as products but are $CD(0,1)$. These spaces show that Lichnerowicz's splitting theorem does not hold for the $CD(0,1)$ condition.

Corollary 2.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a bounded C^2 function which has bounded first and second derivatives. Then there exists λ large enough such that the metric $dt^2 + e^{\frac{2\phi}{n-1}} g_{S^n_\lambda}$ with $f = \phi$ is $CD(0,1)$ where S^n_λ is the sphere of constant Ricci curvature λ.*

Proof. In the calculations above we have $f_L = 1$. Choose λ such that $\lambda \geq \sup_r \left(\frac{1}{n-1}\frac{\partial^2\phi}{\partial r^2} e^{\frac{2\phi}{n-1}}\right)$, then by Proposition 2.3, the desired space is $CD(0,1)$. $\square$

In the next section, we will show that split spaces are the only complete spaces with f bounded above which are $CD(0,1)$ and contain a line.

3. Proof of the splitting theorem

We now turn our attention to proving the splitting theorem. The first component is a Bochner formula. The usual Bochner formula for Ricci curvature is that for a C^3 function h we have

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess} h|^2 + \mathrm{Ric}(\nabla h, \nabla h) + g(\nabla h, \nabla \Delta h).$$

Using Cauchy-Schwarz on the $|\mathrm{Hess} h|^2$ term and assuming the Ricci curvature bound $\mathrm{Ric} \geq K$ gives

$$\frac{1}{2}\Delta|\nabla h|^2 \geq \frac{(\Delta h)^2}{n} + K|\nabla h|^2 + g(\nabla h, \nabla \Delta h).$$

Now let f be a function on M, the weighted, or f-Laplacian is $\Delta_f = \Delta - D_{\nabla f}$. Then one has the following formula, [Lic70]

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\mathrm{Hess} h|^2 + \mathrm{Ric}_f^\infty(\nabla h, \nabla h) + g(\nabla h, \nabla \Delta_f h). \tag{3.1}$$

For curvature dimension inequalities of generalized dimension less than n we have the following Bochner type formula.

Lemma 3.1. *Let (M^n, g, f) be a manifold with density that is $CD(K, n-m)$ for some integer $m = 1, 2, \ldots n$. Suppose that h is a C^3 function in a neighborhood of a point p such that $\mathrm{Hess} h|_p$ has m non-zero eigenvalues. Let $v = e^{f/m}$, then*

$$\frac{1}{2}v^2\Delta_f|\nabla h|^2 \ \geq v^2\frac{(\Delta_f h)^2}{m} + v^2K|\nabla h|^2 + g(\nabla h, \nabla(v^2\Delta_f h))$$

Moreover, equality is achieved if and only if the m non-zero eigenvalues of $\mathrm{Hess} h|_p$ are all equal and $\mathrm{Ric}_f^{n-m}(\nabla h, \nabla h) = K|\nabla h|^2$.

Proof. We start with (3.1) multiplied by v^2,

$$v^2\frac{1}{2}\Delta_f|\nabla h|^2 = v^2|\mathrm{Hess}h|^2 + v^2\mathrm{Ric}_f^{\infty}(\nabla h,\nabla h) + v^2 g(\nabla h, \nabla\Delta_f h).$$

Then we have

$$g(\nabla h, \nabla(v^2\Delta_f h)) = v^2 g(\nabla h, \nabla \Delta_f h) + 2v^2\frac{g(\nabla h,\nabla f)}{m}\Delta_f h$$

and

$$\begin{aligned} v^2|\mathrm{Hess}h|^2 \geq v^2\frac{(\Delta h)^2}{m} &= v^2\frac{(\Delta_f h + g(\nabla h,\nabla f))^2}{m}\\ &= v^2\left(\frac{(\Delta_f h)^2}{m} + 2\frac{g(\nabla h,\nabla f)}{m}\Delta_f h + \frac{g(\nabla h,\nabla f)^2}{m}\right). \end{aligned}$$

Combining these three equations gives

$$\frac{1}{2}v^2\Delta_f|\nabla h|^2 \ \geq v^2\frac{(\Delta_f h)^2}{m} + v^2\mathrm{Ric}_f^{n-m}(\nabla h,\nabla h) + g(\nabla h, \nabla(v^2\Delta_f h)).$$

Applying $\mathrm{Ric}_f^{n-m} \geq K$ then gives the formula in the lemma.

If the inequality is an equality then we must have $\mathrm{Ric}_f^{n-m}(\nabla h,\nabla h) = K|\nabla h|^2$ and $|\mathrm{Hess}h|^2 = \frac{(\Delta h)^2}{m}$, which implies all of the non-zero eigenvalues of Hess$h$ are the same. $\square$

Note that Lemma 3.1 will apply to any function h when $m = n$, thus it gives a Bochner formula for $CD(K, 0)$. In this paper, we'll be applying this to a (generalized) distance function r, i.e. a function such that $|\nabla r| = 1$ on an open set where the function r is smooth. For a distance function, we have $\nabla_{\nabla r}\nabla r = 0$ implying that Hess$r$ has at most $(n-1)$ non-zero eigenvalues and that the integral curves of r are unit speed geodesics. From the Bochner formula we derive a new Laplacian comparison for the distance function for the condition $CD(0, 1)$.

Theorem 3.2. *Let (M, g, f) satisfy the $CD(0, 1)$ condition. Fix a point $p \in M$ and let r be the distance function to p. Let q be a point such that r is smooth at q, and let $\gamma(t)$ be the unique minimal geodesic from p to q, parametrized by arc-length. Then*

$$(\Delta_f r)(q) \leq \frac{(n-1)}{v^2(q)\int_0^{r(q)} v^{-2}(\gamma(t))dt},$$

where $v = e^{\frac{f}{n-1}}$.

Remark 3.3. Note that when f is constant, we have that $v = c$ for a positive constant and

$$v^2(q)\int_0^{r(q)} v^{-2}(\gamma(t))dt = r(q)$$

so we recover the usual Laplacian comparison, $\Delta r \leq \frac{n-1}{r}$ for $\mathrm{Ric} \geq 0$.

Proof. Apply Lemma 3.1 to $h = r$ to obtain

$$\frac{d}{dt}\left(v^2\Delta_f r\right) \leq -v^2\frac{(\Delta_f r)^2}{n-1}.$$

If we set $\lambda = \left(v^2\Delta_f r\right)\circ\gamma$ we have

$$\dot{\lambda} \leq -\frac{\lambda^2}{v^2(n-1)}$$

which is a Ricatti equation that was also used in [Wyl15]. For any sufficiently small ε, we have

$$\begin{aligned} \dot{\lambda} &\leq -\frac{\lambda^2}{v^2(n-1)} \\ (n-1)\int_\varepsilon^{r(q)} \frac{\dot{\lambda}}{\lambda^2}dt &\leq -\int_\varepsilon^{r(q)} v^{-2}(\gamma(t))dt \\ (n-1)\left(-\lambda^{-1}(r(q)) + \lambda^{-1}(\varepsilon)\right) &\leq -\int_\varepsilon^{r(q)} v^{-2}(\gamma(t))dt. \end{aligned}$$

Since $\lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$ we have

$$\begin{aligned} (n-1)\left(-\lambda^{-1}(r(q))\right) &\leq -\int_0^{r(q)} v^{-2}(\gamma(t))dt \\ \lambda(r(q)) &\leq \frac{(n-1)}{\int_0^{r(q)} v^{-2}(\gamma(t))dt}. \end{aligned}$$

This implies the result by the definition of λ. $\square$

The proof of our splitting theorem follows the classical argument using Busemann functions. Given a non-compact manifold M and a ray γ we define the Busemann function to γ to be the function $b^\gamma(x) = \lim_{t\to\infty}(t - d(x,\gamma(t)))$. b^γ is Lipschitz with Lipschitz constant 1 and is thus differentiable almost everywhere. We want to show that when we have a $CD(0,1)$ space with f bounded above, then $\Delta_f b^\gamma \geq 0$.

At the points where the Busemann function is not smooth, we interpret $\Delta_f b^\gamma$ in the weak sense in terms of barrier functions. That is, for a Lipschitz function h we say that $\Delta_f(h) \geq 0$ at a point x if, for every $\varepsilon > 0$, there is a C^2 function h_ε defined in a neighborhood of x such that $h_\varepsilon(x) = b^\gamma(x)$, $h_\varepsilon \leq b^\gamma$ in a neighborhood of x, and $\Delta_f(h_\varepsilon) \geq -\varepsilon$. The notion that a function have $\Delta_f h \leq 0$ is defined similarly. We call the functions h_ε *barrier functions*.

Lemma 3.4. *Suppose that (M,g,f) is $CD(0,1)$ and f is bounded above, then $\Delta_f(b^\gamma) \geq 0$.*

Proof. Let $x \in M$, we construct the barrier functions for b^γ in the standard way. That is, let $t_i \to \infty$ and let σ_i be minimal geodesics from x to $\gamma(t_i)$, the sequence $\sigma'(0)$ sub-converges to some $v \in T_xM$. Let $\overline{\gamma}$ be the geodesic with $\overline{\gamma}(0) = x$ and $\overline{\gamma}'(0) = v$. Then $\overline{\gamma}$ is a ray, called an asymptotic ray to γ.

Define $h_t(y) = t - d(y,\overline{\gamma}(t)) + b^\gamma(x)$, by the standard arguments in for example [WW09], h_t is a smooth barrier function to b^γ at x. Now we compute

$$\Delta_f(h_t) = -\Delta_f(d(y,\overline{\gamma}(t)) \geq \frac{-(n-1)}{v^2(y)\int_0^{d(y,\overline{\gamma}(t))} v^{-2}(\gamma(s))ds}$$

By the assumption that f is bounded from above, we have that the quantity $\int_0^{d(y,\overline{\gamma}(t))} v^{-2}(\gamma(s))ds$ goes to ∞ as $t \to \infty$, implying that $\Delta_f(b^\gamma) \geq 0$. $\square$

Aside from using the Bochner formula to control the Laplacian of the distance function and thus the Busemann functions, the other application of the Bochner formula used in the splitting theorem is in classifying constant gradient harmonic functions on spaces with $\mathrm{Ric} \geq 0$ as linear functions in a flat factor. We get a different rigidity classification for $CD(0,1)$.

Lemma 3.5. *Suppose that (M, g, f) is $CD(0,1)$ where (M, g) is a complete Riemannian manifold. If there is a smooth function r on (M, g) such that $|\nabla r|^2 = 1$ and $\Delta_f(r) = 0$, then the metric g is a warped product of the form $g = dr^2 + e^{\frac{2\phi(r)}{n-1}} g_L$ and $f = \phi(r) + f_L$ where $f_L : L \to \mathbb{R}$.*

Proof. The fact that M splits topologically as $\mathbb{R} \times L$ is a simple consequence of Morse theory and is true whenever one has a smooth function r with $|\nabla r| = 1$. We can write the metric as $g = dr^2 + g_r$, where g_r is the metric restricted to a level set of r. The assumptions imply that we have equality in Lemma 3.1, so $\mathrm{Ric}^1_f(\nabla r, \nabla r) = 0$ and $\mathrm{Hess} r = \alpha g_r$ for some function α. But we also have $\Delta r = (n-1)\alpha = g(\nabla f, \nabla r)$ so

$$\mathrm{Hess} r = \frac{g(\nabla f, \nabla r)}{n-1} g_r.$$

This implies that

$$L_{\nabla r}\left(e^{\frac{-2f}{n-1}} g_r\right) = 0$$

which implies that $g_r = e^{\frac{2(f(r,\cdot)-f(0,\cdot))}{n-1}} g_0$. This gives us that the metric is a twisted product $g = dr^2 + e^{\frac{2f}{n-1}} g_L$ where $g_L = e^{\frac{-2f(0,\cdot)}{n-1}} g_0$ is a fixed metric on L. Proposition 2.2 then implies a warped product splitting, which completes the proof. □

Now with the lemmas above we can quickly prove the splitting theorems using the standard arguments involving Busemann functions.

Proof of Theorem 1.2. Let γ be a line in our space, and let γ^+ and γ^- be the two rays that make up the line γ. Let $b^{\pm}$ be the corresponding Busemann functions. From Lemma 3.4 we know that $\Delta_f(b^{\pm}) \geq 0$. Using the standard arguments in the first part of the proof of, for example, [WW09, Theorem 6.1] using the maximum principle one obtains that $b^+ = -b^-$ thus $\Delta_f(b^{\pm}) = 0$, which implies that $b^{\pm}$ are both smooth by elliptic regularity. An additional standard argument then gives that $|\nabla(b^{\pm})| = 1$ at every point. From Lemma 3.5 we obtain the warped product splitting. □

Proof of Corollary 1.3. Since (M, g, f) is $CD(0, N)$, it is also $CD(0,1)$ so Theorem 1.2 implies that g is a warped product, $g = dr^2 + e^{\frac{2\phi}{n-1}} g_L$ and $f = \phi(r) + f_L$. As we saw in Section 2, we also have that $\mathrm{Ric}^1_f(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$. Then,

$$\begin{aligned}
\mathrm{Ric}^N_f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= \mathrm{Ric}^1_f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + \left(\frac{N-1}{(n-1)(n-N)}\right)\left(\frac{d\phi}{dr}\right)^2 \\
&= \left(\frac{N-1}{(n-1)(n-N)}\right)\left(\frac{d\phi}{dr}\right)^2.
\end{aligned}$$

Since $N - 1 < 0$ we must have $\frac{d\phi}{dr} = 0$. This implies that metric g is a product metric, which we can write as $g = dr^2 + g_L$ and f is a function on L only. □

4. Structure theorem and applications

Now we consider applying our splitting theorem iteratively. First note that if we have a space with a line that is $CD(0,N)$ for $N<1$ with f bounded above, then we have an isometric product splitting $M=\mathbb{R}\times L$ and f is a function on L. Then, (L,g_L,f_L) is $CD(0,N-1)$ and f_L is bounded above, so if L contains a line then we can apply the splitting theorem to L. Iterating this argument, one obtains that M is isometric to a product metric of the form $M=\mathbb{R}^k\times L$ and f is a function on L with (L,g_L,f_L) being $CD(0,N-k)$. A similar argument in the $CD(0,1)$ case yields the following.

Theorem 4.1. *Suppose that (M,g,f) is $CD(0,1)$ and f is bounded above, then M is diffeomorphic to $\mathbb{R}^k\times L$ and the metric g is of the form*

$$g \quad = \quad dr^2+e^{\frac{2\phi(r)}{n-1}}g_{\mathbb{R}^{k-1}}+e^{\frac{2\phi(r)}{n-1}}g_L$$

Where $g_{\mathbb{R}^{k-1}}$ denotes the Euclidean metric, (L,g_L) has no lines, $f=\phi(r)+f_L$, and (L,g_L,f_L) is $CD(0,1-k)$.

Proof. Let (M,g,f) be $CD(0,1)$ with f bounded above and containing a line. Then we have $g=dr^2+e^{\frac{2\phi(r)}{n-1}}g_{L'}$, $f=\phi+f_{L'}$ and by Proposition 2.3 $(L,g_{L'},f_{L'})$ is $CD(0,0)$. Since f is bounded above, so is $f_{L'}$ so we can split L' isometrically as $\mathbb{R}^{k-1}\times L$ with $f_{L'}=f_L$ is a function on L only and (L,g_L,f_L) is $CD(0,1-k)$ and L contains no lines. Then we obtain the splitting

$$g=dr^2+e^{\frac{2\phi(r)}{n-1}}g_{\mathbb{R}^{k-1}}+e^{\frac{2\phi(r)}{n-1}}g_L.$$

□

Despite the ease with which we can prove this structure theorem, there is one subtle point that is important for the applications below. That is, for a warped product, it is not true that lines in the fiber L will always lift to lines in M nor that lines in M always project to lines in L. For a simple example of the later case, consider Euclidean space written in polar coordinates $dr^2+r^2g_{S^{n-1}}$ and a line that is not through the origin. We first show that this issue with projections is excluded if we use the fact again that f is bounded above.

Proposition 4.2. *Consider a warped product metric of the form $g=dr^2+v^2(r)g_L$ where $v>0$ is bounded from above. Let $\gamma:(a,b)\to \mathrm{M}$ be a unit speed minimizing geodesic in M and write $\gamma(s)=(\gamma_1(s),\gamma_2(s))$, where γ_1 and γ_2 are the projections in the factors $\mathbb{R}$ and L. Then*

1. *γ_2 is either constant or its image is a minimizing geodesic in (L,g_L).*
2. *If γ_2 is not constant and γ is a line in M, then the image of γ_2 is a line in L.*

Note that $\gamma_2(s)$ itself will not necessarily be a geodesic because it will not be parametrized with constant speed.

Proof. First we want to show that the image of γ_2 is a length minimizing curve in g_L. To see this, parametrize γ such that $\gamma:[0,1]\to M$, then

$$length(\gamma)=|\dot{\gamma}(t)|=\sqrt{|\dot{\gamma_1}(t)|^2_{g_{\mathbb{R}}}+v^2(\gamma_1(t))|\dot{\gamma_2}(t)|^2_{g_L}}$$

Suppose that $length_{g_L}(\gamma_2(t)) > d(\gamma_2(0), \gamma_2(1))$, and let σ be a minimal geodesic in L from $\gamma_2(0)$ to $\gamma_2(1)$. Then

$$|\dot{\sigma}(t)| = length(\sigma) < length(\gamma_2) = \int_0^1 |\dot{\gamma_2}(t)| dt$$

In particular, there must be an open interval (α, β) with $|\dot{\sigma}(t)| < |\dot{\gamma_2}(t)|$. On (α, β) the curve $\overline{\gamma}(t) = (\gamma_1(t), \sigma(t))$ is clearly shorter than $\gamma|_{(\alpha,\beta)}$, which contradicts the fact that γ is minimizing.

Now assume that γ is a line. From (1), in order to show that γ_2 is a line we just need to show that the length of both branches of $\gamma_2(s)$ as $s \to \infty$ and $s \to -\infty$ are infinite in g_L. To see this we use the geodesic equations for the warped product, from which it follows (see [O'N83, Remark 39, p. 208]) that the quantity $(v \circ \gamma_1)^4 g_L(\dot{\gamma_2}, \dot{\gamma_2}) = C$ for some constant C. Since v is bounded above, this implies that there is a universal constant A not depending on s such that $g_L(\dot{\gamma_2}, \dot{\gamma_2}) \geq A$. This implies that the length of both branches of γ_2 in L is infinite. $\square$

Corollary 4.3. *For the splitting given in Theorem 4.1, any line in (M, g) is constant on the L factor.*

On the other hand, for the metric

$$g = dr^2 + e^{\frac{2\phi(r)}{n-1}} g_{\mathbb{R}^{k-1}} + e^{\frac{2\phi(r)}{n-1}} g_L,$$

the lines in the $\mathbb{R}^{k-1}$ factor will not necessarily lift to lines in M. However, we can avoid this issue if we assume a two-sided bound on f, which will always be satisfied for the universal cover of a compact $CD(0,1)$ space.

Lemma 4.4. *Suppose that (M, g, f) is $CD(0,1)$ with f is bounded (above and below) and contains a line, then either ϕ is constant in Theorem 4.1 or M is diffeomorphic to $\mathbb{R} \times L$ and $g = dr^2 + e^{\frac{2\phi}{n-1}} g_L$ where $f = \phi + f_L$ and (L, g_L, f_L) has $(\mathrm{Ric}_{g_L})^0_{f_L} > 0$. In particular, (L, g_L) does not admit a line.*

Proof. Split $g = dr^2 + e^{\frac{2\phi}{n-1}} g_{L'}$, $f = \phi + f_{L'}$ as in the proof of Theorem 4.1. We claim if ϕ is non-constant then $(\mathrm{Ric}_{g_{L'}})^0_{f_{L'}} > 0$. If $(\mathrm{Ric}_{g_{L'}})^0_{f_{L'}}(V, V)$ was not positive for some choice of V, then by Proposition 2.3, $\frac{\partial^2 \phi}{\partial r^2} \leq 0$. Since f is a bounded function this implies that ϕ is bounded and concave function of r, so it must be constant. $\square$

Now we turn our attention to applications of the splitting theorem to spaces with symmetry and the fundamental group. These come from studying the isometry group of non-compact spaces which are $CD(0,1)$ with f bounded that admit a line.

When ϕ is constant we are in the case considered by Cheeger-Gromoll where we have a product metric $g = g_{\mathbb{R}^k} + g_L$ where L admits no lines. The main observation is that isometries F of g must take lines to lines. This implies that F preserves the distributions tangent to $\mathbb{R}^k$ and L in M. Thus, F splits into $F = F_1 \times F_2$ where $F_1 \in \mathrm{Isom}(\mathbb{R}^k)$ and $F_2 \in \mathrm{Isom}(L, g_L)$.

When ϕ is not constant we obtain a similar result. By Lemma 4.4 and Corollary 4.3 we have $g = dr^2 + e^{\frac{2\phi}{n-1}} g_L$ and the only line for the g metric is the one in the r-direction. Thus, since isometries take lines to lines, we also have that F splits as $F_1 \times F_2$ where $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : L \to L$, moreover, simple calculation shows that

for any isometry of this form for a warped product we must have $F_1 \in \mathrm{Isom}(\mathbb{R})$ with $\phi \circ F_1 = \phi$ and $F_2 \in \mathrm{Isom}(L, g_L)$ (see Exercise 11 on page 214 of [O'N83]).

Now we can apply these results to the universal cover of a compact space which is $CD(0,1)$.

Theorem 4.5. *Let (M, g, f) be compact and $CD(0,1)$, let $(\widetilde{M}, \widetilde{g}, \widetilde{f})$ be the universal cover of M with the covering metric $\widetilde{g}$ and $\widetilde{f}$ the pullback of f to $\widetilde{M}$. Then either*

1. *$\widetilde{M}$ is compact,*
2. *$(\widetilde{M}, \widetilde{g})$ is isometric to a product of a flat metric on $\mathbb{R}^k$ and a compact manifold L.*
3. *$\widetilde{M}$ is diffeomorphic to $\mathbb{R} \times L$ where L is compact and $\tilde{g} = dr^2 + e^{\frac{2\phi}{n-1}} g_L$, $\tilde{f} = \phi + f_L$, and (L, g_L, f_L) is $CD(K, 0)$ for some $K > 0$.*

Note that Case (3) can certainly occur, as a metric of the form $\tilde{g} = dr^2 + e^{\frac{2\phi}{n-1}} g_L$ with ϕ periodic and $f = \phi$ will cover a $CD(0,1)$ metric on $S^1 \times L$,

Proof. Assume (1) is not true so that $\widetilde{M}$ is non-compact. A standard argument shows that $\widetilde{M}$ contains a line. To see this take a ray γ in M and let $t_i \to \infty$. Then, since the deck transformations of $\widetilde{M}$ act by isometries of $\widetilde{g}$ there is a compact set K (e.g. a fundamental domain) and a sequence of isometries F_i such that $F_i(\gamma(t_i)) \in K$ for all i. Let p be a limit of a convergent subsequence of the $F_i(\gamma(t_i))$. For some further subsequence we also have $DF_i(\dot{\gamma}(t_i))$ converging to a unit vector $v \in T_pM$. Let σ be the geodesic with $\sigma(0) = p$ and $\dot{\sigma}(0) = v$ then, since the distance that a geodesic minimizes is continuous with respect to its initial conditions, σ is a line.

We can then split $\tilde{g} = dr^2 + e^{\frac{2\phi}{n-1}} g_L$, $\tilde{f} = \phi + f_L$. If ϕ is constant, then we can split M into $\mathbb{R}^k \times L$ where L contains no lines. If L is non-compact, then the argument above, using the fact that the isometries of $\mathbb{R}^k \times L$ must split, would produce a line in L, therefore L must be compact in this case, and we obtain (2).

Now suppose that ϕ is not constant. We need to show that L is compact. By Lemma 4.4 L does not contain any lines. The idea is to assume that L is non-compact and argue by contradiction that L must then contain a line. This is complicated by the fact that geodesics of L do not necessarily lift to geodesics of $\widetilde{M}$, so we must use the geodesic equations of a warped product again.

Fix $p \in L$ and let x_j be a sequence going off to infinity in L. Let γ^j be a unit speed minimal geodesic in $\widetilde{M}$ from $(0, p)$ to $(0, x_j)$. Write $\gamma^j(t) = (\gamma_1^j(t), \gamma_2^j(t))$, using the warped product geodesic equations again as above we see there is a constant C_j such that $e^{\frac{2\phi(\gamma_1^j(t))}{n-1}} |\dot{\gamma}_2^j(t)|_{g_L} = C_j$. Since $\gamma_1 : (a, b) \to \mathbb{R}$ must have $\gamma_1(a) = \gamma_1(b) = 0$, it must have a critical point, t_0. At that point,

$$1 = |\dot{\gamma}^j|_g = e^{\frac{\phi(\gamma_1^j(t_0))}{n-1}} |\dot{\gamma}_2^j(t_0)|_{g_L}$$

so $C_j = e^{\frac{\phi(\gamma_1^j(t_0))}{n-1}}$. Since ϕ is bounded, this implies that C_j is bounded. Then there is a positive constant A such that $|\dot{\gamma}_2^j(t)|_{g_L} \geq A$ for all j and t.

Now consider γ a ray which is a sub-sequential limit of γ^j. Write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Since $|\dot{\gamma}_2^j(t)|_{g_L} \geq A$ we have that $|\dot{\gamma}_2(0)|_{g_L} \geq A$. Using the geodesic equations for a warped product in the same way as above, we obtain a possibly different A such that $|\dot{\gamma}_2(t)|_{g_L} \geq A$ for all t. Now take a sequence $\gamma(t_i)$ with $t_i \to \infty$

and pull back γ by isometries F^i to produce a line σ as before. Since each F^i splits as a map $F^i_1 \times F^i_2$ where F^i_2 is an isometry of L, if we write $\sigma(s) = (\sigma_1(s), \sigma_2(s))$ then $|\dot{\sigma}_2(s)|_{g_L} \geq A$ for all s. Therefore σ_2 is not a constant map so by Proposition 4.2 the image of σ_2 forms a line in L which achieves the desired contradiction. $\square$

We now use Theorem 4.5 to prove Theorem 1.5.

Proof of Theorem 1.5. First we show (3). From the proof of Theorem 1.2 using Lemma 3.5, if $\widetilde{M}$ contains a line, then at every point p there is a vector $V \in T_pM$ with $\mathrm{Ric}^1_f(V,V) = 0$. Thus by Theorem 4.5, if $\mathrm{Ric}^1_f > 0$ at a point, then $\widetilde{M}$ must be compact and $\pi_1(M)$ is finite.

Identify $\pi_1(M)$ as a subgroup of the isometries of $\widetilde{M}$ acting properly discontinuously and freely on $\widetilde{M}$. Then as we discuss above, for $F \in \pi_1(M)$ we can write $F = F_1 \times F_2$ where $F_2 \in \mathrm{Isom}(L, g_L)$ and, by Theorem 4.5, F_1 is in the isometry group of flat $\mathbb{R}^k$ ($k = 1$ when ϕ is non-constant). The projection of $\pi_1(M)$ into each factor then produces a short exact sequence

$$0 \to E \to \pi_1(M) \to \Gamma \to 0$$

Where Γ is a crystallographic group, i.e. a discrete, cocompact subgroup of the isometry group of $\mathbb{R}^k$ and E is a finite group. By [Wil00, Theorem 2.1] $\pi_1(M)$ is then the fundamental group of a compact manifold of nonnegative sectional curvature.

Finally if $b_1(M) = n$ then k must be n and then $\widetilde{M}$ must be flat, implying that M is also flat. $\square$

Finally we show that ϕ must be constant when M is locally homogeneous.

Proof of Theorem 1.6. Let $\widetilde{M}$ be the universal cover of a compact locally homogeneous space M. Then $\widetilde{M}$ is homogeneous. Apply Theorem 4.5 and suppose that ϕ were not constant. Then, since the isometry group splits and acts transitively, between any two points in $\mathbb{R}$ there must be a reflection or translation F_1 such that $\phi = \phi \circ F_1$. This implies that ϕ must be constant. Then from Theorem 4.5 $\widetilde{M} = \mathbb{R}^k \times L$ where L is compact and homogeneous. However, by [KW, Proposition 3.7], L must then also have non-negative Ricci curvature. Then M has non-negative Ricci curvature. The rest of the structure then follows from [CG71, Theorem 5]. $\square$

5. Manifolds with boundary

In this section, we prove a version of the splitting theorem for compact manifolds with boundary. The boundary ∂M is also assumed to be smooth with outward unit normal ν. Let H be the mean curvature of ∂M with respect to the outward normal vector. The weighted (or generalized) mean curvature of the boundary is $H_f = H - g(\nabla f, \nu)$. Just as the usual mean curvature arises in the first variation of the Riemannian volume, the weighted mean curvature arises in the first variation of the measure $e^{-f}d\mathrm{vol}_g$.

We have the following splitting phenomenon.

Theorem 5.1. *Suppose that (M, g, f) is a compact manifold with boundary which is $CD(0,1)$, if $H_f \geq 0$ (M is generalized mean convex) and M has more than one boundary component, then M is a warped product over an interval $M = [a,b] \times L$,*

$f = \phi(r) + f_L(x)$, where $\phi : [a, b] \to \mathbb{R}$ and $f_L : L \to \mathbb{R}$, and $g_M = dr^2 + e^{\frac{2\phi(r)}{n-1}} g_L$ for a fixed metric g_L on L.

The warped products in the conclusion of the theorem have $H_f \equiv 0$ on ∂M and $\mathrm{Ric}_f^1\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0$, so we have the following corollary.

Corollary 5.2. *Suppose that (M, g, f) is a compact manifold with boundary which is $CD(0,1)$ and M is generalized mean convex. If $\mathrm{Ric}_f > 0$ at a point in the interior of M, or $H_f > 0$ at a point in ∂M, then M has only one boundary component.*

By the same argument as in Corollary 1.3 we also have an isometric product in the conclusion of Theorem 5.1 if (M, g, f) is $CD(0, N)$ for $N < 1$. Using the ideas in the next section, Theorem 5.1 can also be extended to non-gradient fields, we leave the statement to the interested reader.

Proof of Theorem 5.1. Let L be a boundary component of M and let r be the distance to L. Let γ be a unit speed geodesic from L to a point $x \in M$ which minimizes the distance from x to L and such that $\gamma(t)$, $t > 0$ is contained in the interior of M. Then applying Lemma 3.1 to r gives

$$D_{\nabla r}(v^2 \Delta_f r(\gamma(r))) \leq 0$$

Moreover, we have that $\Delta_f r(\gamma(r)) \to -H_f(\gamma(0))$ as $r \to 0$, so we have that $\Delta_f r \leq 0$ along the geodesic γ.

Now let L_1 be a component of ∂M. Let L_2 be another boundary component which minimizes the distance from L_1 to L_2 among the other boundary components. let r_1, r_2 be the distance functions to L_1 and L_2 respectively. Consider the function $e(x) = r_1(x) + r_2(x)$. By the triangle inequality $e(x) \geq d(L_1, L_2)$ and the points where the minimum is achieved must lie on a geodesic γ which connects L_1 and L_2 and only touches ∂M at its endpoints. By the argument above $\Delta_f e = \Delta_f r_1 + \Delta_f r_2 \leq 0$ at such a minimal point. This implies that e must be constant by the strong maximum principle. Then there is a constant a so that $r_1 = a - r_2$, which implies that $\Delta_f r_1 = 0$. By elliptic regularity this shows that r_1 is smooth on the interior of M. Then r_1 is a smooth function with $|\nabla r_1|^2 = 1$ and $\Delta_f(r_1) = 0$. The argument in Lemma 3.5 then shows that M is a warped product. □

6. Non-gradient Vector fields

In this section we explain how the results above also have versions for non-gradient potential fields. Curvature dimension inequalities have a well known definition for vector fields.

Definition 6.1. Let X be a vector field on a Riemannian metric (M^n, g). The N-dimensional generalized Ricci tensor is

$$\mathrm{Ric}_X^N = \mathrm{Ric} + \frac{1}{2} L_X g - \frac{X^\sharp \otimes X^\sharp}{N - n}$$

where $L_X g$ is the Lie derivative of g with respect to X and $X^\sharp$ is the dual one form of X coming from g. We say that (M, g, X) is $\mathrm{CD}(\lambda, N)$, $(\lambda \in \mathbb{R}, N \in (-\infty, \infty])$ if $\mathrm{Ric}_X^N \geq \lambda$.

Note that with this definition $\mathrm{Ric}_f = \mathrm{Ric}_{\nabla f}$, so the results of this section should be viewed as generalizing our results in section 3 to non-gradient fields.

All of our results in the gradient case involve bounds on the potential function f. While there is no potential function for a non-gradient field, we can still make sense of bounds by integrating X along geodesics. Let X be a vector field on a Riemannian manifold (M, g). Let $\gamma : (a, b) \to M$ be a geodesic that is parametrized by arc-length. Define

$$f_\gamma(t) = \int_a^t g(\dot{\gamma}(s), X(\gamma(s)))ds$$

f_γ is a real valued function on the interval (a, b) with the property that $\dot{f}_\gamma(t) = g(\dot{\gamma}(t), X(\gamma(t)))$. When $X = \nabla f$ is a gradient field then $f_\gamma = f(\gamma(t)) - f(\gamma(a))$, in the non-gradient case we think of f_γ as being the anti-derivative of X along the curve γ. We now introduce the condition we will need for results in this section.

Definition 6.2. Let (M, g) be a smooth non-compact complete Riemannian manifold with a smooth vector field X. Then we say (M, g, X) is X-*complete* if for every point $y \in M$

$$\limsup_{r\to\infty} \inf_{l(\gamma)=r} \left\{ \int_0^r e^{\frac{-2f_\gamma(\gamma(s))}{n-1}} ds \right\} = \infty.$$

where the infimum is taken over all minimizing unit speed geodesics γ of the metric g with $\gamma(0) = y$. If $X = \nabla f$ we say that (M, g, f) is f-complete.

In general, f_γ depends on the parametrization of γ only up to an additive constant, so the notion of X-completeness does not depend on the parametrization of the geodesic. Also note that if a vector field X has the property that f_γ is bounded for all unit speed minimizing geodesics then it is X-complete. However, even in the gradient case, f-completeness is a weaker condition than f bounded above.

One way to interpret f-completeness is that the quantity $\int_0^r e^{\frac{-2f_\gamma(\gamma(s))}{n-1}} ds$ is, up to a multiplicative factor, the energy of the curve γ in the conformal metric $e^{\frac{-2f}{(n-1)}} g$. From this we can see that f-completeness implies that $e^{\frac{-2f}{(n-1)}} g$ is a complete metric. Alternately, X-completeness is equivalent to the completeness of a certain modified affine connection, see [WY] for more details.

Our most general splitting theorem is the following.

Theorem 6.3. *Let (M, g) be a complete Riemannian metric supporting a vector field X which is $CD(0, 1)$ and X-complete. If (M, g) admits a line then M is a twisted product metric on $\mathbb{R} \times L$. If $X = \nabla f$ then M is a warped product.*

On the other hand, it is easy to see from the formulas in Proposition 2.1 that we can not obtain a warped product splitting for non-gradient fields.

Proposition 6.4. *There are metrics of the form $dr^2 + e^{\frac{2\phi}{n-1}} g_{S^n}$ which are $CD(0, 1)$ where ϕ is not a function of r and X is not gradient.*

Proof. For any function ϕ, let $X = \frac{2}{n-1}\frac{\partial \phi}{\partial r} + \left(\frac{n-3}{n-1}\right) \nabla \phi$. Note that X is a gradient field if and only if ϕ is a function of r. Then a calculation using Proposition 2.1 shows that $\mathrm{Ric}^1_X \left(\frac{\partial}{\partial r}, Y\right) = 0$ for all Y. The formula for Ric^1_X for vectors tangent to S^n is much more complicated. However, it is of the form

$\mathrm{Ric}^1_X(U, V) = \mathrm{Ric}^{S^n}(U, V) +$ terms involving ϕ and its first and second partial derivatives.

The terms on the right will also go to zero as ϕ and its partial derivatives go to zero. Therefore, if we take g_{S^n} to be a round sphere with positive Einstein constant

λ, there is a constant A which depends on λ and the dimension such that if ϕ and its first and second derivatives are all less than A, then $\mathrm{Ric}^1_X(U,V) \geq 0$. $\square$

Now we turn our attention to proving the splitting theorem. The first component is the Bochner formula applied to the twisted Laplacian $\Delta_X = \Delta - D_X$, of the distance function, which follows from the same argument as in Lemma 3.1.

Lemma 6.5. *Suppose (M^n, g, X) is CD(0,1) and that r is a smooth distance function on an open subset of a Riemannian manifold (M,g). Let γ be an integral curve of r and let $v_\gamma = e^{\frac{f_\gamma}{n-1}}$. Then,*

$$\frac{d}{dr}\left(v_\gamma^2 \Delta_X r\right) \leq -v_\gamma^2 \frac{(\Delta_X r)^2}{n-1}$$

where $\frac{d}{dr}$ denotes the derivative along γ. Moreover, if equality is achieved at a point p then the $(n-1)$ non-zero eigenvalues of $\mathrm{Hessr}|_p$ are all equal and $\mathrm{Ric}^1_X(\nabla r, \nabla r) = 0$ at p.

Proof. As is well known, the usual Bochner formula for functions,

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}h|^2 + \mathrm{Ric}(\nabla h, \nabla h) + g(\nabla h, \nabla \Delta h),$$

can also be modified for non-gradient fields in the same manner as in (3.1) to

$$\frac{1}{2}\Delta_X|\nabla h|^2 = |\mathrm{Hess}h|^2 + \mathrm{Ric}^\infty_X(\nabla h, \nabla h) + g(\nabla h, \nabla \Delta_X h).$$

This follows directly from the identity

$$\begin{aligned} g(\nabla h, \nabla(D_X h)) &= D_{\nabla h} g(X, \nabla h) \\ &= g(\nabla_{\nabla h} X, \nabla h) + g(X, \nabla_{\nabla h} \nabla h) \\ &= \frac{1}{2}\left(L_X g(\nabla h, \nabla h) + D_X |\nabla h|^2\right) \end{aligned}$$

The proof is then identical to the proof of Lemma 3.1 using v_γ in the place of v in the argument. $\square$

Following the same arguments as in Section 3, it then follows that if (M,g,X) is $CD(0,1)$ and X-complete then $\Delta_X(b_\gamma) \geq 0$ for any Busemann function. In the gradient case, we then have the splitting theorem when (M,g,f) is f-complete. The other element needed for the splitting theorem in the non-gradient case is a generalization of Lemma 3.5 to the non-gradient case.

Lemma 6.6. *Suppose that (M,g) is a complete Riemannian manifold with a smooth vector field X that is $CD(0,1)$. If there is a smooth function r on (M,g) such that $|\nabla r|^2 = 1$ and $\Delta_X r = 0$, then*

1. *M splits topologically as $\mathbb{R} \times N$, with metric of the form $g = dr^2 + e^{\frac{2\phi}{n-1}} g_N$, where g_N is a metric on N and $\phi : M \to \mathbb{R}$.*
2. *$\mathrm{Ric}^1_X(\nabla r, \nabla r) = 0$.*
3. *$X = \frac{\partial \phi}{\partial r}\frac{\partial}{\partial r} + U$ where $U \perp \frac{\partial}{\partial r}$.*

Proof. Since $|\nabla r| = 1$ we have $\mathbb{R} \times N$ topologically and $g = dr^2 + g_r$, where g_r is the metric restricted to a level set of r. In terms of this splitting write the vector field $X = a(r,x)\frac{\partial}{\partial r} + Y$ where $a : M \to \mathbb{R}$ and Y is tangent to N at every point. Then,

for γ an integral curve of r, we have $g(X,\dot{\gamma}) = a$. Define a function ϕ globally on M via the formula

$$\phi(r,x) = \int_0^r a(t,x)dt.$$

ϕ is clearly a smooth function since a is smooth.

The assumptions imply that we have equality in Lemma 6.5, so $\mathrm{Ric}_X^1(\nabla r, \nabla r) = 0$ and $\mathrm{Hess}r = \alpha g_r$ for some function α. But we also have $\Delta r = (n-1)\alpha = g(X, \nabla r)$ so

$$\mathrm{Hess}r = \frac{g(X,\nabla r)}{n-1} g_r$$

Since $D_{\nabla r}\phi = a(r,x) = g(X,\nabla r)$ this implies that

$$L_{\nabla r}\left(e^{\frac{-2\phi}{n-1}} g_r\right) = 0$$

which implies that $g_r = e^{\frac{2(\phi(r,\cdot)-\phi(0,\cdot))}{n-1}} g_0$. This gives us that the metric is a twisted product $g = dr^2 + e^{\frac{2\phi}{n-1}} g_N$ where $g_N = e^{\frac{-2\phi(0,\cdot)}{n-1}} g_0$ is a fixed metric on N. Note that the function ϕ automatically satisfies (3) as $\frac{\partial \phi}{\partial r} = a(r,x)$. $\square$

The proof of Theorem 6.3 then follows using Lemma 6.6 and the same arguments as in Section 3. In the $CD(0,N)$ case $N<1$ we also obtain the isometric product splitting.

Corollary 6.7. *Suppose that (M,g) is a complete Riemannian manifold and X is a smooth vector field on M which is X-complete and $CD(0,N)$ for $N<1$. If (M,g) admits a line then M is isometric to a product metric $M = \mathbb{R}\times L$ and X is a vector field on L.*

Proof. Since (M,g,X) is $CD(0,N)$, it is also $CD(0,1)$ so Theorem 6.3 implies that g is a twisted product, $g = dr^2 + e^{\frac{2\phi}{n-1}} g_L$. We also have that $\mathrm{Ric}_X^1(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 0$. Since $X = \frac{\partial \phi}{\partial r}\frac{\partial}{\partial r} + U$ where $U \perp \frac{\partial}{\partial r}$, this gives us

$$\mathrm{Ric}_X^L\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -\left(\frac{1-N}{(n-1)(n-N)}\right)\left(\frac{\partial \phi}{\partial r}\right)^2$$

so we must have $\frac{\partial \phi}{\partial r} = 0$. This implies that metric g is a product metric, which we can write as $g = dr^2 + h_L$, where h_L is a conformal metric to g_L.

We can also show X is a vector field on L only using the fact that $\mathrm{Ric}_X^1\left(\frac{\partial}{\partial r}, V\right) = 0$ for all $V \perp \frac{\partial}{\partial r}$. To see this, fix a point x in N, let $\frac{\partial}{\partial y^i}$, $i = 1, \ldots, n-1$ be an orthonormal basis of local coordinates around x in the g_L metric. Write $X = b_i \frac{\partial}{\partial y^i}$ Then

$$0 = \mathrm{Ric}_X^1\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial y^k}\right) = \frac{1}{2}\frac{\partial b_k}{\partial r}$$

So X is a vector field on L that does not depend on r. $\square$

We would also like to generalize Theorem 4.1 to the case where $X = \nabla f$, but the upper bound on f is replaced by f-completeness. The only obstacle that arises in the proof is that when a space splits, it is not a priori clear that f-completeness on the whole space should imply f-completeness on the fiber. It turns out, however, that we can use geodesic equations for a warped product to show that f-completeness does have this natural property for the spaces in our splitting theorem.

Consider a split space to be a warped product of the form $dr^2 + e^{\frac{2\phi}{n-1}} g_L$ with potential function $f = \phi(r) + f_L$. Let γ be a unit speed minimizing geodesic and write $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, where γ_1 and γ_2 are the projections in the factors $\mathbb{R}$ and L. The f-completeness condition implies for any ray of (M, g_M) that $\int_0^\infty \left(e^{\frac{-2\phi(\gamma_1(s))}{n-1}} e^{\frac{-2f_L(\gamma_2(s))}{n-1}} \right) ds$ diverges. We have the following proposition.

Proposition 6.8. *Suppose M is a split space that is f-complete and let $\gamma : (a, b) \to M$ be a minimizing geodesic of the form $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ then*

(1) *If γ_2 is not constant and γ is a line in M, then the image of γ_2 is a line in L.*
(2) *The manifold with density (L, g_L, f_L) is f_L-complete.*

Proof. Assume that γ is a line. From part (1) of Proposition 4.2, which is true for any warped product, in order to show that γ_2 is a line we just need to show that the length of both branches of $\gamma_2(s)$ as $s \to \infty$ and $s \to -\infty$ are infinite in g_L. From the geodesic equations for the warped product, we have $e^{\frac{4\phi(\gamma_1(s))}{n-1}} g_L(\dot{\gamma_2}, \dot{\gamma_2}) = C$ for some constant C. Then

$$length(\gamma_2|_{[0,\infty)}) = \int_0^\infty |\dot{\gamma_2}|_{g_L} ds = C \int_0^\infty e^{\frac{-2\phi(\gamma_1(s))}{n-1}} ds.$$

Assume for contradiction that γ_2 had finite length in L. Then the function $e^{\frac{-2f_L(\gamma_2(s))}{n-1}}$ is uniformly bounded in s and so the f-completeness assumption applied to γ implies that $\int_0^\infty e^{\frac{-2\phi(\gamma_1(s))}{n-1}} ds$ is infinite. Up to a constant this is the length of γ_2, so we obtain a contradiction. The same argument also shows that the length of the branch of γ_2 with $s \to -\infty$ is also infinite.

In order to show (2), fix a point $p \in L$ and let $\beta(\tau)$ be a unit speed geodesic in L with $\beta(0) = p$ which is minimizing for $\tau \in [0, r]$. We want to estimate $\int_0^r e^{\frac{-2f_L(\beta(\tau))}{n-1}} d\tau$. First note that from part (1) of Proposition 4.2 and the uniqueness of minimizing geodesics that there is a geodesic in M which is of the form $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ $s \in (0, t)$ such that the image of γ_2 is the image of β.

We again apply the warped product geodesic equations to the geodesic γ to see that there is a constant $C_\gamma \neq 0$ such that $|\dot{\gamma_2}|_{g_L} = C_\gamma e^{\frac{-2\phi(\gamma_1(s))}{n-1}}$. By compactness, we can thus choose C_1, C_2 uniformly so that $C_1 e^{\frac{-2\phi(\gamma_1(s))}{n-1}} \leq |\dot{\gamma_2}|_{g_L} \leq C_2 e^{\frac{-2\phi(\gamma_1(s))}{n-1}}$ for every unit speed geodesic β with $\beta(0) = p$.

$$\begin{aligned} \int_0^t \left(e^{\frac{-2\phi(\gamma_1(s))}{n-1}} e^{\frac{-2f_L(\gamma_2(s))}{n-1}} \right) ds &\leq \frac{1}{C_1} \int_0^t e^{\frac{-2f_L(\gamma_2(s))}{n-1}} |\dot{\gamma_2}|_{g_L} ds \\ &= \frac{1}{C_1} \int_0^r e^{\frac{-2f_L(\beta(\tau))}{n-1}} d\tau \end{aligned} \tag{6.1}$$

where in the last line, we have re-parametrized the curve γ_2 by arc-length in g_L to obtain β. Moreover,

$$r = length(\beta) = length(\gamma_2) \leq C_2 \int_0^t e^{\frac{-2\phi(\gamma_1(s))}{n-1}} ds \tag{6.2}$$

where t is the length in M of the geodesic γ.

(6.2) implies that as $r \to \infty$, $t \to \infty$. Then f-completeness of M implies that the left hand side of (6.1) blows up as $r \to \infty$ and thus so does the right hand side, showing that L is f_L-complete. $\square$

On the other hand, in the non-gradient setting, we do not have the applications of the splitting theorem to the universal cover of a compact manifold because it is not true that a vector field lifted to the universal cover is X-complete. The question of the extent to which the Theorems 1.5 and 1.6 are true for non-gradient fields appears to be largely open.

References

[Bak94] Dominique Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, Lectures on probability theory (Saint-Flour, 1992), Lecture Notes in Math., vol. 1581, Springer, Berlin, 1994, pp. 1–114 (French).

[BÉ85] D. Bakry and Michel Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206 (French).

[Bor74] Christer Borell, *Convex measures on locally convex spaces*, Ark. Mat. **12** (1974), 239–252.

[BL76] Herm Jan Brascamp and Elliott H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Analysis **22** (1976), no. 4, 366–389.

[CG71] Jeff Cheeger and Detlef Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128.

[FLZ09] Fuquan Fang, Xiang-Dong Li, and Zhenlei Zhang, *Two generalizations of Cheeger-Gromoll splitting theorem via Bakry-Émery Ricci curvature*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 563–573.

[FLGRKÜ01] M. Fernández-López, E. García-Río, D. N. Kupeli, and B. Ünal, *A curvature condition for a twisted product to be a warped product*, Manuscripta Math. **106** (2001), no. 2, 213–217.

[Gig14] Nicola Gigli, *An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature*, Anal. Geom. Metr. Spaces **2** (2014), 169–213.

[Gig] ———, *The splitting theorem in non-smooth context.* arXiv:1302.5555.

[KW] Lee Kennard and William Wylie, *Positive weighted sectional curvature*, Indiana Math J. To appear, arXiv:1410.1558.

[Kla] Bo'az Klartag, *Needle decompositions in Riemannian geometry*, Mem. Amer. Math. Soc. To appear, arXiv:1408.6322.

[KM] Alexander V. Kolesnikov and Emanuel Milman, *Poincaré and Brunn-Minowski inequalities on weighted Riemannian manifolds with boundary.* arXiv:1310.2526.

[Lic70] André Lichnerowicz, *Variétés riemanniennes à tenseur C non négatif*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A650–A653 (French).

[Lic71] ———, *Variétés kählériennes à première classe de Chern non negative et variétés riemanniennes à courbure de Ricci généralisée non negative*, J. Differential Geometry **6** (1971/72), 47–94 (French).

[Mila] Emanuel Milman, *Beyond traditional Curvature-Dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension*, Trans. Amer. Math. Soc. arXiv:1409.4109.

[Milb] ———, *Harmonic Measures on the Sphere via Curvature-Dimension*, Ann. Fac. Sci. Toulouse Math. arXiv:1505.04335.

[Oht] Shin-ichi Ohta, *(K, N)-convexity and the curvature-dimension condition for negative N*, J. Geom. Anal. arXiv:1310.7993.

[OT11] Shin-ichi Ohta and Asuka Takatsu, *Displacement convexity of generalized relative entropies*, Adv. Math. **228** (2011), no. 3, 1742–1787.

[OT13] Shin-Ichi Ohta and Asuka Takatsu, *Displacement convexity of generalized relative entropies. II*, Comm. Anal. Geom. **21** (2013), no. 4, 687–785.

[O'N83] Barrett O'Neill, *Semi-Riemannian geometry,* Pure and Applied Mathematics, vol. 103, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[Wil00] Burkhard Wilking, *On fundamental groups of manifolds of nonnegative curvature*, Differential Geom. Appl. **13** (2000), no. 2, 129–165.

[WW09] Guofang Wei and Will Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405.

[WW16] Eric Woolgar and William Wylie, *Cosmological singularity theorems and splitting theorems for N-Bakry-Émery spacetimes*, J. Math. Phys. **57** (2016), no. 2.

[Wyl15] William Wylie, *Sectional curvature for Riemannian manifolds with density*, Geom. Dedicata **178** (2015), 151–169.

[WY] William Wylie and Dmytro Yeroshkin, *On the geometry of Riemannian manifolds with density*. arXiv:1602.08000.

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